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ON CIRCLE-VALUED COCYCLES OF AN ERGODIC MEASURE-PRESERVING TRANSFORMATION†

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ABSTRACT

Analytic necessary and sufficient conditions are given for a circle-valued function f to generate a cocycle which is a multiple of a coboundary. These conditions are then used to derive some other new criteria for cocycles to be coboundaries.

1. Introduction

If G is a group, X is a space on which G acts (on the right by $x \to x \,.\, g$), and H is another group, we say that R is a *Cocycle* of this action of G, with *Coefficients* in the group H, if R is a function from $X \times G$ into H satisfying the "cocycle identity": $R(x, gg') = R(x, g)R(x \,.\, g, g')$.

Of particular interest are the cocycles which are coboundaries. A cocycle R is a *Coboundary* if there exists a function β from X into H such that $R(x, g) = \beta(x)[\beta(x \,.\, g)]^{-1}$. Given a cocycle R, it is virtually impossible in practice to tell *a priori* whether it is a coboundary. It is the purpose of this paper to develop some analytic techniques for identifying coboundaries among cocycles.

Let us take as the group G the group $\mathbf{Z}$ of integers, in which case the action of G on X is determined by a single transformation τ of X. As a primary example, one on which we will test all our theorems, let X be the half-open interval $[0, 1)$, let θ be an irrational number, and define τ on X by $\tau(x) = x + \theta \bmod 1$. Let us

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take for the coefficient group H the circle group $\mathbf{T}$. A cocycle R for this simple case is then a function from $X \times \mathbf{Z}$ into $\mathbf{T}$ satisfying:

$$R(x, n + m) = R(x, n)R((x + n\theta), m),$$

from which it follows that the entire function R is determined by the single function $f(x) = R(x, 1)$. Indeed, for $n > 0$, we have

$$R(x, n) = f(x)f(x + \theta)f(x + 2\theta)\cdots f(x + (n - 1)\theta).$$

Further, R is a coboundary if and only if there exists a function $g: X \to \mathbf{T}$ such that $f(x) = g(x)/g(x + \theta)$, which reduces the relatively abstract questions about coboundaries to more concrete questions about circle-valued functions on the interval.

Whether a given function f determines a cocycle which is a coboundary remains as a largely unsolved problem even in this simplest case. Veech in [9] [10] and [11], Petersen in [7], Stewart in [8], Merrill in [4], and others have studied f's which are two-valued step functions on X. Merrill also obtained in [4] some corresponding results on general step functions, and in [1] f's of the form $f(x) = e^{2\pi isx}$ are treated.

Section 2 contains the main theorems (2.3, 2.4, and 2.5) analytically characterizing coboundaries, the technical cornerstone in the author's opinion being Lemma 2.2. Section 3 includes some applications. We prove that the function $f(x)f(x + \theta)f(x + 2\theta)\cdots f(x + (k - 1)\theta)$ is a multiple of a coboundary for translation by $k\theta$ if and only if f itself is a multiple of a coboundary for translation by θ, apparently a new result. Finally, we generalize a result of Merrill which characterizes multiples of coboundaries as those functions f for which $f(x)/f(x + t)$ is a coboundary for all t.

2. Circle-valued cocycles for a single measure-preserving transformation

Let X be a space, let μ be a probability measure on X, and let τ be an invertible, ergodic, μ-preserving transformation on X. If f is a measurable function from X into the circle $\mathbf{T}$, we say that f is a circle-valued *Coboundary* of τ if there exists a measurable $g: X \to \mathbf{T}$ such that

$$f(x) = g(x)/g(\tau(x))$$

for μ almost all x. In this case, we say that f is the *Coboundary of* g, and we write $f = dg$. If f and f' are measurable functions from X into $\mathbf{T}$, we say that f is *cohomologous* to f' if f/f' is a coboundary.

We say that f is a *Projective Coboundary* if there exists a $g: X \to \mathbf{T}$ and a scalar λ of modulus 1 such that

$$f(x) = \lambda g(x)/g(\tau(x))$$

for μ almost all x.

REMARK. The set of all coboundaries for τ forms a group under pointwise multiplication, as does the set of all projective coboundaries.

We let U_τ be the unitary operator on $L^2(X, \mu)$ defined by $[U_\tau(g)](x) = g(\tau(x))$.

Given f and τ as above, we define U_f by $[U_f(g)](x) = f(x)[U_\tau(g)](x) = f(x)g(\tau(x))$.

2.1. PROPOSITION. *The function f is a projective coboundary if and only if the operator U_f has nontrivial discrete spectrum. And, f is a coboundary if and only if U_f has an eigenvalue in common with U_τ*

PROOF. It follows from ergodicity of τ, and the fact that $|f(x)| = 1$, that any eigenfunction g, belonging to an eigenvalue λ for U_f, is of constant nonzero absolute value, whence it can be taken to have unit modulus. But then f is the projective coboundary $\lambda g(x)/g(\tau(x))$. The converse is obvious.

Assuming $f(x) = \lambda g(x)/g(\tau(x))$, then if f is a coboundary, the constant function λ is a coboundary for τ. But if $\lambda = h(x)/h(\tau(x))$, then h is an eigenfunction for U_τ with eigenvalue $1/\lambda$. Since the set of eigenvalues for the unitary operator U_τ necessarily forms a subgroup of $\mathbf{T}$, it follows that λ is also an eigenvalue for U_τ. Q.E.D.

We introduce the following definition, a generalization of the one given in the introduction.

DEFINITION. Let $f: X \to \mathbf{T}$. For $n > 0$, define $f^{(n)}$ on X by $f^{(n)}(x) = f(x)f(\tau(x)) \cdots f(\tau^{n-1}(x))$, and set $f^{(0)} \equiv 1$.

2.2. LEMMA. *For any measurable $f: X \to \mathbf{T}$ we have*

$$\lim_{N \to \infty} (1/N) \sum_{n=0}^{N-1} \int f^{(n)}(x) d\mu(x)$$

exists, is real, and is nonnegative.

PROOF. Let $f_N = (1/N) \sum_{n=0}^{N-1} f^{(n)}$. Then $f_N \in L^2(\mu)$ and $\| f_N \|_2 \leqq 1$. The lemma follows immediately if $\{f_N\}$ tends weakly to 0 in L^2 as N tends to ∞.

Suppose then that h is a nonzero weak cluster point of $\{f_N\}$. Then the function $h \circ \tau = h/f$. By ergodicity, h has nonzero constant modulus, and $f(x) = h(x)/h(\tau(x))$. But then

$$f^{(n)}(x) = h(x)/h(\tau^n(x)) \quad \text{and} \quad f_N(x) = h(x)(1/N) \sum_{n=0}^{N-1} (1/h(\tau^n(x)).$$

This implies, by the L^2 Ergodic Theorem, that $\lim_{N\to\infty} f_N$ is the function $h \int (1/h(x))d\mu(x)$ in L^2. Therefore,

$$\lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \int f^{(n)}(x)d\mu(x) = \lim_{N\to\infty} \int f_N(x)d\mu(x)$$

$$= \int h(x)d\mu(x) \, . \int (1/h(x))d\mu(x) = \left| \int h \right|^2 \Big/ \| h \|^2 \geqq 0,$$

and this completes the proof.

2.3. Theorem. *Let $f: X \to \mathbf{T}$. Then f is a projective coboundary if there exists some sequence $\{a_n\}$ of complex numbers of modulus $\leqq 1$ such that* $\{(1/N) \sum_{n=0}^{N-1} (a_n \int f^{(n)}(x)d\mu(x))\}$ does not tend to 0.

Proof. Assume the existence of such a sequence $\{a_n\}$. Then clearly $\limsup_{N\to\infty} (1/N) \sum_{n=0}^{N-1} | \int f^{(n)}d\mu(x)| > 0$. This implies that

$$\limsup_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \left| \int f^{(n)}(x)d\mu(x) \right|^2 > 0.$$

For if the former lim sup $= \varepsilon > 0$, then there must exist a set S of nonnegative integers having positive density for which $| \int f^{(n)}(x)d\mu(x)| > \varepsilon$ for $n \in S$. But then, for each $n \in S$, $| \int f^{(n)}(x)d(x)|^2 > \varepsilon^2$, whence the latter lim sup must be positive.

Now $\int f^{(n)}(x)d\mu(x) = ((U_f)^n(1), 1) = p(n) = \hat{\nu}(n)$, where p is the positive definite function on $\mathbf{Z}$ associated to the unitary operator U_f and the constant function 1, and where ν is the probability measure on $\mathbf{T}$ whose Fourier transform is p. We have

$$0 < \limsup_{N\to\infty} (1/N) \sum_{n=0}^{N-1} |\hat{\nu}(n)|^2 = \limsup_{N\to\infty} (1/N) \sum_{n=0}^{N-1} (\nu * \tilde{\nu})^\wedge(n)$$

($\tilde{\nu}$ being the measure on $\mathbf{T}$ defined by $\tilde{\nu}(E) = \nu(\bar{E})$)

$$= \limsup_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \int \alpha^{-n} d(\nu * \tilde{\nu})(\alpha)$$

$$= \limsup_{N\to\infty} \int [(1/N)(1-\alpha^{-n})/(1-\alpha^{-1})] d(\nu * \tilde{\nu})(\alpha).$$

Since this integrand is uniformly bounded in α, and tends to 0 except at the point $\alpha = 1$, it must be that $\{1\}$ has positive measure under $\nu * \tilde{\nu}$. But

$$\nu * \tilde{\nu}(\{1\}) = \int \chi_{\{1\}}(\alpha\beta) d\nu(\alpha) d\tilde{\nu}(\beta) = \int \nu(\alpha) d\nu(\alpha).$$

Therefore, the measure ν gives positive mass to some point $\alpha \in \mathbf{T}$. It follows then from spectral theory that the operator U_f has some discrete spectrum. Then, by Proposition 2.1, f is a projective coboundary as desired.

Remark. An obvious corollary to the preceding theorem is that f is a projective coboundary if there exists an $\varepsilon > 0$ and a set S of positive density such that $|\int f^{(n)}| > \varepsilon$ for all $n \in S$.

Another consequence is the following:

Corollary. *A measurable $f: X \to \mathbf{T}$ is a projective coboundary if and only if there exists a $\lambda \in \mathbf{T}$ and a measurable $\psi: X \to \mathbf{T}$ such that*

$$\lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \lambda^n \int (d\psi)^{(n)}(x) f^{(n)}(x) d\mu(x) > 0.$$

Proof. If $f = \gamma dg$, set $\lambda = 1/\gamma$ and $\psi = 1/g$. Conversely, given λ and ψ, the theorem implies that $(d\psi)f$ is a projective coboundary, whence so is f.

Something even more precise can be said when the spectrum of U_τ is purely discrete.

2.4. Theorem. *Suppose that the operator U_τ has purely discrete spectrum. Then the following are equivalent for a measurable function $f: X \to \mathbf{T}$.*

(i) *f is a projective coboundary.*

(ii) *For some $\lambda \in \mathbf{T}$ the sequence $\{(1/N) \sum_{n=0}^{N-1} (\lambda^n \int f^{(n)}(x) d\mu(x))\}$ does not tend to 0.*

(iii) *For some* $\lambda \in \mathbf{T}$ *the sequence* $\{(1/N)\sum_{n=0}^{N-1}(\lambda^n f^{(n)})\}$ *does not tend to* 0 *in* L^2.

PROOF. Denote an orthonormal set of eigenfunctions for U_τ by $\{\phi_j\}$, and let λ_j be the eigenvalue associated to ϕ_j. By ergodicity, we have that $|\phi_j(x)| \equiv 1$. Suppose f is a projective coboundary, say $f(x) = \gamma g(x)/g(\tau(x))$, and choose a j for which $\int g(x)\phi_j(x)d\mu(x) = c_j \neq 0$. Note also that $d_j = \int (1/g\phi_j) \neq 0$. Define $g' = g\phi_j$. Then $f(x) = \gamma\lambda_j g'(x)/g'(\tau(x))$. Let $\lambda = 1/(\gamma\lambda_j)$. Then $\lambda^n f^{(n)}(x) = g'(x)/g'(\tau^n(x))$. So, the sequence $\{(1/N)\sum_{n=0}^{N-1}(\lambda^n f^{(n)})\}$ has limit in L^2 the function $g'(x)\int(1/g'(x))d\mu(x) = g'(x)d_j$, and this is nonzero in L^2. This shows that (i) implies (iii). Since $\int g'(x)d_j d\mu(x) = c_j d_j$, we also have that (i) implies (ii). By Theorem 2.3, (ii) implies (i). Finally, if the sequence $\{(1/N)\sum_{n=0}^{N-1}(\lambda^n f^{(n)})\}$ has nonzero limit h in L^2, then as before we see that $h(\tau(x)) = h(x)/\lambda f(x)$, whence f is a projective coboundary, and (iii) implies (i).

REMARK. Properties (ii) and (iii) are actually equivalent for any ergodic and μ-preserving transformation τ. One shows easily that not (iii) implies not (ii) and not (ii) implies not (iii). But (i) need not imply (ii) or (iii) when τ (U_τ) has some continuous spectrum, e.g., when τ is a Bernoulli shift. Indeed, if $g : X \to \mathbf{T}$ is nonzero and orthogonal to every eigenfunction for U_τ, and if f is defined to be the coboundary of g, i.e., $f(x) = g(x)/g(\tau(x))$, then for any $\lambda \in \mathbf{T}$ we have that the limit of the sequence $\{(1/N)\sum_{n=0}^{N-1}(\lambda^n f^{(n)})\}$ is the limit of $\{(1/N)\sum_{n=0}^{N-1}(\lambda^n g(x)/g(\tau^n(x)))\}$, which is 0 by the Ergodic Theorem.

2.5. THEOREM. *Suppose* U_τ *has purely discrete spectrum. Then* f *is a coboundary if and only if there exists an eigenvalue* λ_j *of* U_τ *such that the sequence* $\{((1/N)\sum_{n=0}^{N-1}(\lambda_j)^n f^{(n)})\}$ *has a nonzero* L^2 *limit* h. *Furthermore*:

(i) $f(x) = 1/\lambda_j)h(x)/h(\tau(x))$.

(ii) $\int h(x)d\mu(x) > 0$.

(iii) $\int h(x)d\mu(x) = \| h \|_\infty^2$.

PROOF. If f is a coboundary, then the proof of (i) implies (iii) in the preceding theorem shows the existence of the required eigenvalue λ_j. Conversely, if $\{(1/N)\sum_{n=0}^{N-1}((\lambda_j)^n f^{(n)})\}$ has a nonzero L^2 limit h, then the proof of (iii) implies (i) in the preceding theorem shows that $f(x) = (1/\lambda_j)h(x)/h(\tau(x))$. But $1/\lambda_j$ is also an eigenvalue of U_τ, whence the constant function $1/\lambda_j$ is the coboundary of the corresponding eigenfunction $\overline{\phi_j}$. Hence f is the product of two coboundaries and is therefore a coboundary itself.

Now, if h is the limit of $\{(1/N)\sum_{n=0}^{N-1}((\lambda_j)^n f^{(n)})\}$, and if $h \neq 0$, then we have seen in the proof of Lemma 2.2 that this limit is

$$h(x) \,.\, \int (1/h) = h(x) \int \bar{h}(x) d\mu(x) / \| h \|_\infty^2,$$

whence $\int h = \int h(x)d\mu(x)[\int \bar{h}(x)d\mu(x)/ \| h \|_\infty^2] = | \int h |^2 / \| h \|_\infty^2$ which shows that $\int h = \| h \|_\infty^2$. Properties (ii) and (iii) follow from this observation, and the proof is complete.

We equip the set $\mathbf{M}$ of all measurable functions from X into $\mathbf{T}$ with the Polish topology of convergence in μ-measure.

2.6. THEOREM. *The set* $\mathbf{P}$ *of all projective coboundaries is a Borel subset of* $\mathbf{M}$, *and there exists a Borel map (cross-section) C from* $\mathbf{P}$ *into* $\mathbf{M} \times \mathbf{T}$ *such that if* $C(f) = (g, \lambda)$, *then* $f = (1/\lambda)dg$.

PROOF. The map $(g, \lambda) \to \lambda dg$ is a continuous homomorphism of $\mathbf{M} \times \mathbf{T}$ into $\mathbf{M}$. The kernel K of this homomorphism is the set of all (g, λ) for which $\lambda dg = 1$, i.e., $g(\tau(x)) = \lambda g(x)$ a.e. For (g, λ) to belong to K, it is necessary and sufficient that g be a multiple of some eigenfunction ϕ_j for U_τ and λ must be the corresponding eigenvalue λ_j. So, K is the set of all pairs $(z\phi_j, \lambda_j)$, for $z \in \mathbf{T}$. Hence, since the set of eigenfunctions for U_τ is in any case an orthonormal set in L^2, K is closed in $\mathbf{M} \times \mathbf{T}$, and the Polish group $(\mathbf{M} \times \mathbf{T})/K$ maps continuously and 1–1 onto the subset $\mathbf{P}$ of the Polish group $\mathbf{M}$. Hence, by the isomorphism theorem (see [3]), the set $\mathbf{P}$ is a Borel set in $\mathbf{M}$, and the inverse map of $\mathbf{P}$ onto $(\mathbf{M} \times \mathbf{T})/K$ is a Borel map. Composing this inverse with a Borel cross-section of $(\mathbf{M} \times \mathbf{T})/K$ into $\mathbf{M} \times \mathbf{T}$ ([6]) gives the desired map C.

COROLLARY. *The set* $\mathbf{P}_0$ *of all coboundaries is a Borel subset of* $\mathbf{M}$, *and there exists a Borel map* C_0 *of* $\mathbf{P}_0$ *into* $\mathbf{M}$ *such that if* $C_0(f) = g$, *then* $f = dg$.

The proof is completely analagous to the one above. We use this corollary in the proof of Theorem 3.2 below.

REMARK. The proofs break down if τ is not measure-preserving. For then the eigenfunctions may not form an orthonormal set (or even be square-integrable), and the subgroup K may not be closed in $\mathbf{M}$.

3. Applications

If f is a projective coboundary for τ, then $f(x)f(\tau(x))$ is a projective coboundary for τ^2 whether τ^2 is ergodic or not (and it can be either way). That the converse is true even when τ^2 is ergodic is not quite so obvious, though it can be shown directly. For exponents k larger than 2, we know of no elementary proof for the following:

3.1. THEOREM. *Assume k is a positive integer for which τ^k is ergodic. Then, a measurable function $f: X \to \mathbf{T}$ is a projective coboundary for τ if and only if $f^{(k)}$ is a projective coboundary for τ^k.*

PROOF. If f is a projective coboundary for τ, say $f(x) = \lambda g(x)/g(\tau(x))$, then $f^{(k)}(x) = \lambda^k g(x)/g(\tau^k(x))$, and so is a projective coboundary for τ^k.

Conversely, suppose that $f^{(k)}$ is a projective coboundary for τ^k. It will suffice to show that there exists a function $\psi: X \to \mathbf{T}$ such that $(d\psi)f$ is a projective coboundary for τ. By assumption, there exists a η and a $\lambda \in \mathbf{T}$ such that $f^{(k)}(x) = \lambda\eta(x)/\eta(\tau^k(x))$. Set $\psi = 1/\eta$. Define a_m to be 0 unless m is a multiple of k, and set $a_{nk} = 1/\lambda^n$. Then:

$$\lim_{N\to\infty} (1/N) \sum_{m=0}^{N-1} a_m \int ((d\psi)f)^{(m)}(x)d\mu(x)$$

$$= \lim_{N\to\infty} (1/Nk) \sum_{m=0}^{Nk-1} a_m \int ((d\psi)f)^{(m)}(x)d\mu(x)$$

$$= \lim_{N\to\infty} (1/Nk) \sum_{n=0}^{N-1} (1/\lambda^n) \int (d\psi)^{(nk)}(x)\lambda^n(\eta/\eta\circ\tau^k)^{(n)}(x)d\mu(x)$$

$$= 1/k > 0.$$

So, by Theorem 2.3, $(d\psi)f$ is a projective coboundary, as desired.

As an example of this theorem, take X to be the interval $[0, 1)$ and τ to be translation mod 1 by an irrational number θ. Take k to be 3. The theorem above asserts that if $f: [0, 1) \to \mathbf{T}$ is measurable and satisfies

$$f(x)f(x+\theta)f(x+2\varphi) = \lambda g(x)/g(x+3\theta),$$

then f must satisfy $f(x) = \lambda' h(x)/h(x+\theta)$. We do not know of an elementary proof of this.

Next, we study how a function's being a projective coboundary is related to its being cohomologous to translates of itself. The theorem below is a generalization of an unpublished result due to K. Merrill.

3.2. THEOREM. *Let Γ be a locally compact group of μ-preserving transformations of X, let H be a closed co-compact subgroup of Γ for which X is homeomorphic to the coset space Γ/H, and assume that for each $x \in X$ there exists a measurable map c_x of X into Γ such that the transformation $y \to [c_x(y)](x)$ is μ-preserving. Suppose finally that τ commutes with each $\gamma \in \Gamma$. Let*

$f: X \to \mathbf{T}$. Then f satisfies $f(x)/f(\gamma(x))$ is a multiplicative coboundary for each $\gamma \in \Gamma$ if and only if f is a projective coboundary.

PROOF. We normalize Haar measure on Γ and H so that $\int_\Gamma f(\gamma)d\gamma = \int_X \int_H f(\gamma h)dhd\mu(\gamma H)$. By the corollary to Theorem 2.6, there exists a Borel map $\gamma \to g_\gamma$ of Γ into $\mathbf{M}$ satisfying $g_\gamma(x)/g_\gamma(\gamma(x)) = f(x)/f(\gamma(x))$ a.e. μ. Again letting $\{\psi_j\}$ denote a countable dense subset of $\mathbf{M}$, it follows, from the corollary to Theorem 2.3, that there exists an integer j and a set S of positive Haar measure in Γ for which

$$\lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \int (d\psi_j)^{(n)}(x)(f/f\circ\gamma)^{(n)}(x)d\mu(x) > 0 \qquad \text{for every } \gamma \in S.$$

It follows then from Fubini's Theorem and Lemma 2.2 that there exists an $x \in X$ such that

$$\begin{aligned}\lim_{N\to\infty} & (1/N) \sum_{n=0}^{N-1} (d\psi_j)^{(n)}(x)f^{(n)}(x) \int_\Gamma (1/f\circ\gamma)^{(n)}(x)d\gamma \\ &= \lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} a_n \int_\Gamma \bar{f}^{(n)}(\gamma(x))d\gamma > 0,\end{aligned}$$

since τ commutes with each γ and where $a_n = (d\psi_j)^{(n)}(x)f^{(n)}(x)$. So,

$$0 < \lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} a_n \int_X \int_H \bar{f}^{(n)}(\gamma(h(x)))dhd\mu(\gamma H).$$

Hence, there exists an $h \in H$ such that

$$\begin{aligned}0 &< \lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} a_n \int_X \bar{f}^{(n)}(c_{h(x)}(y)(h(x)))d\mu(y) \\ &= \lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} a_n \int_X \bar{f}^{(n)}(y)d\mu(y),\end{aligned}$$

which implies, by Theorem 2.3, that $\bar{f}$ and hence also f is a multiplicative projective coboundary. Q.E.D.

COROLLARY. *Suppose X is a compact abelian group, μ is Haar measure on X, α is an element of X for which the cyclic subgroup $\mathbf{Z}\alpha$ is dense in X, and τ is the transformation on X given by $\tau(x) = x\alpha$. For $f: X \to \mathbf{T}$, we have that f is a projective coboundary if and only if f is cohomologous to each of its translates $f \circ \sigma$, where $f \circ \sigma(x) = f(\sigma x)$, for $\sigma \in X$.*

PROOF. We remark that τ is μ-ergodic and μ-preserving, and that it is sufficient to define $c_x(y) = yx^{-1}$.

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